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A COMMON FIXED POINT THEOREM BY USING THE CONCEPT OF GENERALIZED CONTRACTION MAP

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ABSTRACT

In the present paper, we obtain a unique common fixed point theorem for three self -maps on complete metric space satisfying a new contraction condition which significantly covers the result of Banach [2] (see also [5], [11]).

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KEYWORDS: Common Fixed Point, Complete Metric Space, Contraction Mapping

1. INTRODUCTION

The Banach contraction principle is one of the most important tools in the field of fixed point theory with reference to the metric space. Theorems related to existence and uniqueness of fixed points are known as fixed point theorems. The theory of fixed points has been become an important tool in non linear functional analysis since 1930. The significance of this field lies in its vast applicability to many branches of mathematics and other sciences. The study of common fixed points of mappings satisfying different contractive conditions has been explored extensively by many mathematicians. Ciric [5], Fisher [6] (see also [7], [8]), Kannan [11], Iseki et al. [9], Pant [8], Pathak et al. [9], Cho et al. [3], Jungck [10] and Stojakovic [17], Powar et al. [14] (see also [15], [16]), Babu et al.[1], Choudhary [4] etc. established some interesting results on common fixed point theorem considering commuting mappings on different metric spaces.

In the present paper, we have assured the existence of common fixed point for three self mappings under the generalized concept of contraction. It is interesting to note that the Banach contraction principle is a special case of proposed generalized contraction.

2. PRELIMINARIES

In order to establish our result, we require the following definitions

Definition 2.1[2] Let (X, d) be a metric space. A mapping $T: X \to X$ is called a **contraction mapping** if there exists a real number k, $0 < k < 1$, such that

$$d(Tx, Ty) \leq k d(x, y), \quad \text{for all } x, y \text{ in } X.$$

Definition 2.2 Let (X, d) be a metric space and let E, F and T be three self mappings defined over X. A point $p \in X$ is called common fixed point of mappings E, F and if Ep = Fp = Tp = p.

Definition 2.3 Let (X, d) be a metric space and let T be a self mappings defined over X. Let $x \in X$ then $O(x) = \{T^n(x) : n = 0, 1, 2, \ldots\}$ is called the orbit of x.

Definition 2.4 A Space X is said to be T- orbitally complete iff every Cauchy sequence which is contained in O(x) for some x in X converges in X.

3. RESULTS ALREADY PROVED

The well-known Banach contraction principle is given below:

Theorem 3.1 [2] If T is a mapping of a complete metric space X into itself such that

$d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in X$ and $0 < \alpha < 1$.Then T has a unique fixed point.

Kannan [11] in 1968 established the following result:

Theorem 3.2 If T is a mapping of a complete metric space X into itself such that $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$ and $0 < \beta < 1/2$.

Then T has a unique fixed point.

Theorem 3.3 If T a self mapping Ton a metric space X satisfying the following property:

$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx)+ cd(y, Ty)+e[d(x, Ty) + d(y, Tx)]$, for all $x, y \in X$, where $a, b, c, e \geq 0$, with $a+b+c+2e < 1$, then T has a unique fixed point provided that X is T- orbitally complete.

4. MAIN RESULTS

In this section, we state and prove our main result.

Theorem4.1 Let (X, d) be a complete metric space and let E, F and T be three continuous self mappings defined over X satisfying the following conditions:

(a) $ET = TE$, $FT=TF$, $E(X) \subset T(X)$, $F(X) \subset T(X)$ (4.1)

$$\text{(b) } d(Ex, Fy) \leq \alpha \frac{d(Tx,Fy)d(Ty,Ex)d(Tx,Ex)d(Tx,Ty)+d(Tx,Fy)}{1+d(Tx,Ey)d(Ty,Ex)d(Tx,Ex)d(Tx,Ty)}$$

$$+ \beta\left[d(Tx,Ex)+d(Ty,Fy)\right]$$

$$+ \gamma\left[d(Tx,Fy)+d(Ty,Ex)\right] + \delta d(Tx,Ty) \tag{4.2}$$

for all $x, y \in X$ with $\alpha, \beta, \gamma, \delta$ are non-negative real numbers such that $2\alpha+2\beta+2\gamma+\delta < 1$.

Then E, F and T have a **unique common fixed point**.

Proof Let x_o be an arbitrary point of X and $\{Tx_n\}$ be a sequence of points of X such that

$Tx_{2n+1} = Ex_{2n}$ and $Tx_{2n+2} = Fx_{2n+1}$ for all n= 0, 1, 2,… (4.3)

It is possible because $E(X) \subset T(X)$ and $F(X) \subset T(X)$

Applying (4.2), we arrive at

$$d(Tx_{2n+1},Tx_{2n+2}) = d(Ex_{2n},Fx_{2n+1})$$

$$\leq \alpha \frac{d(Tx_{2n},Fx_{2n+1})\, d(Tx_{2n+1},Ex_{2n})\, d(Tx_{2n},Tx_{2n+1}) d(Tx_{n},Ex_{2n})+d(Tx_{2n},Fx_{2n+1})}{1+d(Tx_{2n},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n})d(Tx_{2n},Tx_{2n+1})d(Tx_{2n},Ex_{2n})}$$

$$+ \beta \left[d\left(Tx_{2n}, Ex_{2n}\right) + d\left(Tx_{2n+1}, Fx_{2n+1}\right) \right]$$

$$+ \gamma \left[d\left(Tx_{2n}, Fx_{2n+1}\right) + d\left(Tx_{2n+1}, Ex_{2n}\right) \right]$$

$$+ \delta\, d\left(Tx_{2n}, Tx_{2n+1}\right) \tag{4.4}$$

In view of (4.3), (4.4) reduce to

$$d\left(Tx_{2n+1}, Tx_{2n+2}\right) \le \alpha\, d\left(Tx_{2n}, Tx_{2n+2}\right)$$

$$+ \beta \left[d\left(Tx_{2n}, Tx_{2n+1}\right) + d\left(Tx_{2n+1}, Tx_{2n+2}\right) \right]$$

$$+ \gamma\, d\left(Tx_{2n}, Tx_{2n+2}\right) + \delta\, d\left(Tx_{2n}, Tx_{2n+1}\right)$$

Applying triangle inequality, we obtain

$$\left(1-\alpha-\beta-\gamma\right) d\left(Tx_{2n+1}, Tx_{2n+2}\right) \le \left(\alpha+\beta+\gamma+\delta\right) d\left(Tx_{2n}, Tx_{2n+1}\right)$$

$$\Rightarrow \qquad d\left(Tx_{2n+1}, Tx_{2n+2}\right) \le \frac{\alpha+\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\, d\left(Tx_{2n}, Tx_{2n+1}\right)$$

Hence $d\left(Tx_{2n+1}, Tx_{2n+2}\right) \le u\, d\left(Tx_{2n}, Tx_{2n+1}\right)$,

where $u = \dfrac{\alpha+\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$ (By hypothesis).

Similarly, we have $d\left(Tx_{2n}, Tx_{2n+1}\right) \le u\, d\left(Tx_{2n-1}, Tx_{n}\right)$

Proceeding in this way, we get

$$d\left(Tx_{2n+1}, Tx_{2n+2}\right) \le u^{2n+1} d\left(Tx_{o}, Tx_{1}\right).$$

Also for any positive integer k > n, it may be written directly that

$$d\left(Tx_{n}, Tx_{n+k}\right) \le \sum_{i=1}^{k} d\left(Tx_{n+i-1}, Tx_{n+i}\right)$$

$$\le \sum_{i=1}^{k} u^{n+i-1} d\left(Tx_{o}, Tx_{1}\right)$$

$$\le \frac{u^{n}}{1-u} d\left(Tx_{o}, Tx_{1}\right) \longrightarrow 0, \text{ as } n \to \infty.$$

Hence $\{Tx_n\}$ is a Cauchy sequence of points of X. Since X is complete, $\{Tx_n\}$ converges to p, for some $p \in X$.

In view of (4.3), both the sequences $\{Ex_{2n}\}$ and $\{Fx_{2n+1}\}$ also converge to p.

Applying the continuity of E, F and T, we conclude

$E(Tx_{2n}) \to Ep$

and $F(Tx_{2n+1}) \to Fp$, as $n \to \infty$. (4.5)

Referring (4.1), we have $E(Tx_{2n}) = TEx_{2n}$ and $F(Tx_{2n+1}) = T\,F\,x_{2n+1}$, for all n=0, 1, 2,…

Letting limit as $n \to \infty$, we have

Ep = Tp = Fp and (4.6)

T(Tp) = T(Ep) = E(Tp) = FE (4.7)

For Ep $\neq$ F(Ep) and referring (4.2), (4.6) and (4.7), we get

$$d(Ep, FEp) \le \alpha \frac{d(Tp, FEp)d(TEp, Ep)d(Tp, TEp)\cdot d(Tp, Ep) + d(Tp, FEp)}{1 + d(Tp, FEp)d(TEp, Ep)d(Tp, TEp)d(Tp, Ep)}$$

$$+ \beta\left[d(Tp, Ep) + d(TEp, FEp)\right]$$

$$+ \gamma\left[d(Tp, FEp) + d(TEp, Ep)\right]$$

$$+ \delta\, d(Fp, TEp)]$$

$$\le (\alpha + 2\gamma + \delta)\, d(Ep, FEp)$$

This implies that $d(Ep, F\,Ep) < d(Ep, F\,Ep)$

which is a contradiction to our assumption F Ep $\neq$ Ep.

Hence, F Ep = Ep .

Thus, we have Ep = F Ep = T Ep = E Ep, when we appeal to (4.7).

This shows that Ep is the common fixed point of mappings E, F and T.

Claim: Common fixed point is unique.

Let, if possible, there exist points q are r in X, with q $\neq$ r, such that

Eq = Fq = Tq = q and Er = Fr = Tr = r

Now d (q, r) = d (Eq, Fr)

$$\le (\alpha + 2\gamma + \delta)\, d(Eq, Er)$$

$\le$ d (q, r)

which leads to a contradiction, hence q = r .

This implies that E, F and T have a **unique common fixed point.**

CONCLUTIONS

It may be observed that the special choices of self maps E, F and T along with the particular values of α, β, γ and δ our result covers the following theorems as special cases:

- Considering E= F, T= I and $\alpha = \beta = \gamma = 0$, we get theorem 3.1(cf.[2]) of section 3.
- Assuming E=F, T=I and $\alpha = \gamma = \delta = 0$, we obtain theorem 3.2 (cf.[11]) of section 3.
- Choosing E = F, T = I and $\alpha = 0$, $\beta = \frac{ud(Tx,Ex)+vd(Ty,Fy)}{1+d(Tx,Ex)+d(Ty,Fy)}$ (where u, v are non

  negative real numbers such that $u+v+2\gamma+\delta < 1$) and considering X is T-orbitally

complete, we get theorem 3.3 (cf. [5]) of section 3.

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